11/03/2021 Balls in Jars Problem

	red	white
Jar 1	3	3
Jar 2	4	2

$P(J_1) = P(J_2) = \frac{1}{2}$

Let J_1 be jar 1 and J_2 be jar 2.
Let R be red and W be white ball.

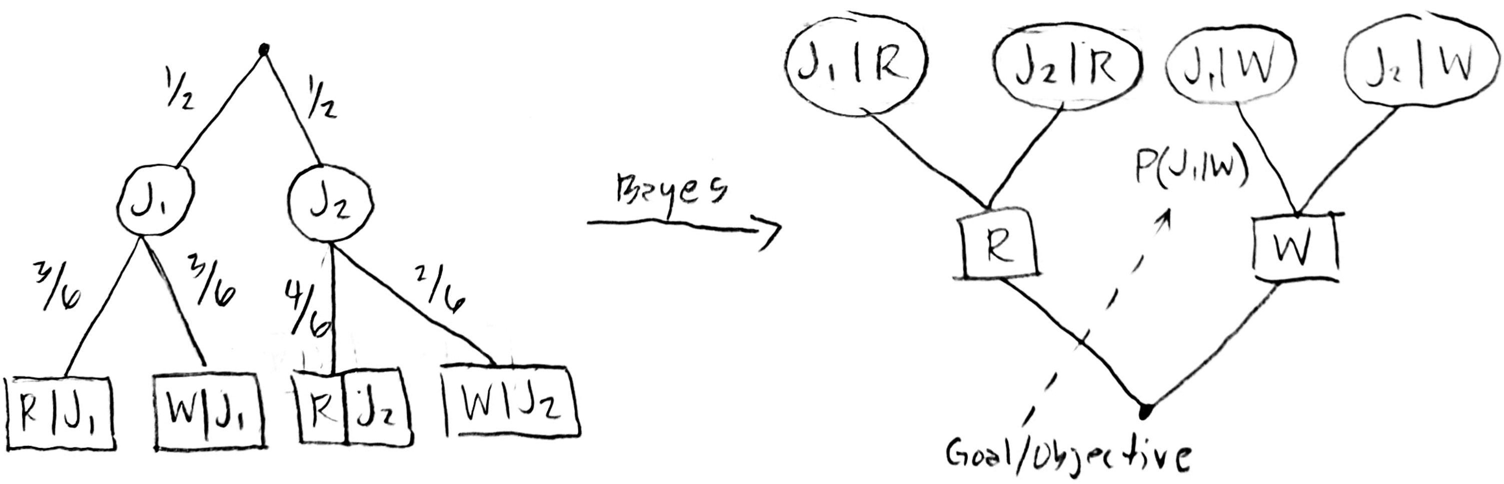
(a) objective $\longrightarrow$

$$\overbrace{P(J_1|W)}^{\text{Posterior}} = \frac{\overbrace{P(J_1)}^{\text{prior}}\ \overbrace{P(W|J_1)}^{\text{likelihood}}}{\underbrace{P(W)}_{\text{marginalization}}}$$

observing a white ball

$$= \frac{P(J_1)P(W|J_1)}{\underbrace{P(J_1)P(W|J_1) + P(J_2)P(W|J_2)}_{\text{law of total probability}}}$$

Visualizing Bayes using trees:



Solution: $P(J_1|W) = \dfrac{(\frac{1}{2})(\frac{3}{6})}{(\frac{1}{2})(\frac{3}{6}) + (\frac{1}{2})(\frac{2}{6})} = \boxed{\frac{3}{5}}$

(b) objective $\longrightarrow$ $P(J_2 \mid R_2) = \dfrac{P(J_2)P(R_2 \mid J_2)}{P(R_2)}$

observing two red balls

$$= \frac{P(J_2)P(R_2 \mid J_2)}{P(J_1)P(R_2 \mid J_1) + P(J_2)P(R_2 \mid J_2)}$$

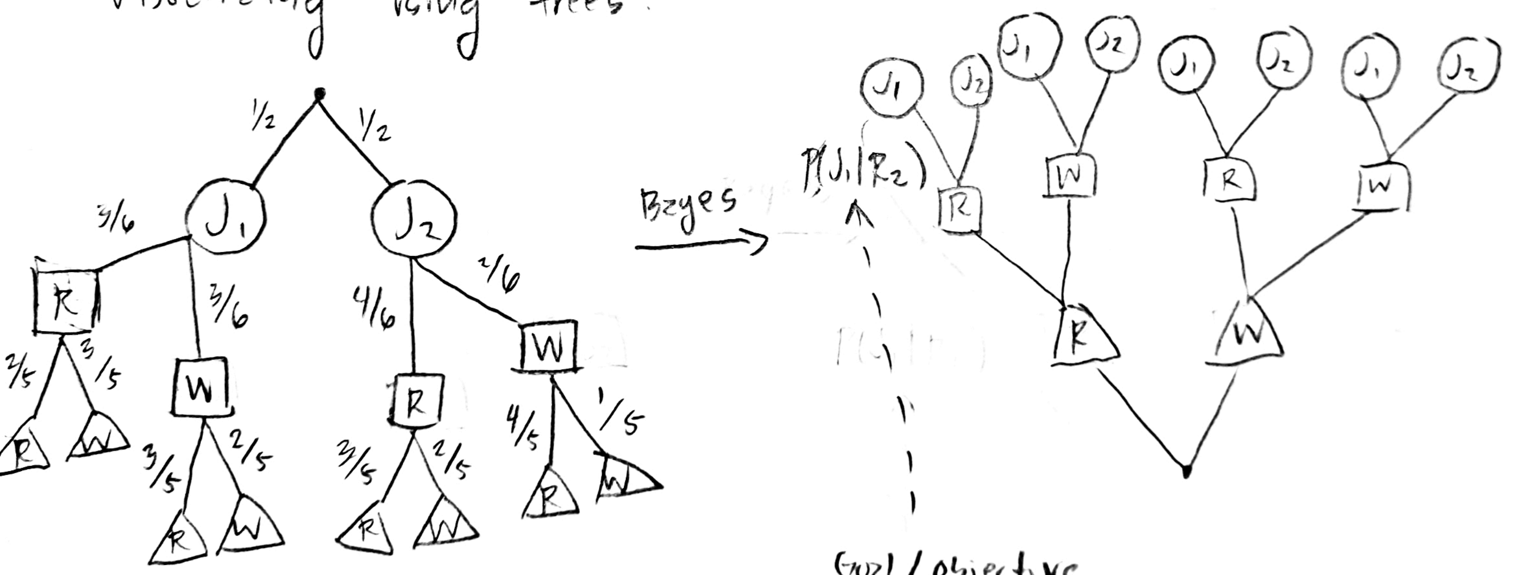
$$P(J_2)P(R_2 \mid J_2) = \left(\tfrac{1}{2}\right)\left(\tfrac{4}{6}\right)\left(\tfrac{3}{5}\right)$$

$$P(J_1)P(R_2 \mid J_1) = \left(\tfrac{1}{2}\right)\left(\tfrac{3}{6}\right)\left(\tfrac{2}{5}\right)$$

$$P(R_2) = P(J_2)P(R_2 \mid J_2) + P(J_1)P(R_2 \mid J_1)$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{4}{6}\right)\left(\tfrac{3}{5}\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{3}{6}\right)\left(\tfrac{2}{5}\right)$$

Visualizing using trees:



Goal / objective

Solution:

$$P(J_2 \mid R_2) = \frac{\left(\frac{1}{2}\right)\left(\frac{4}{6}\right)\left(\frac{3}{5}\right)}{\left(\frac{1}{2}\right)\left(\frac{4}{6}\right)\left(\frac{3}{5}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{6}\right)\left(\frac{2}{5}\right)} = \boxed{\frac{2}{3}}$$

(c)

objective $\longrightarrow P(J_2 | R_2 W) = \dfrac{P(J_2)\,P(R_2W | J_2)}{P(R_2W)}$

observing two reds and one white

Visualizing the tree would take a lot of space and time. However, we can be clever and use the choose function.

How many ways can I pick 2 red balls from 3 balls?

$${}_3C_2 = \binom{3}{2} = 3 \text{ ways}$$

Considering J_1, $P(J_1)P(R_2W | J_1)$

$$= P(J_1)P(RRW | J_1) \rightarrow \left(\tfrac{3}{6}\right)\left(\tfrac{2}{5}\right)\left(\tfrac{3}{4}\right) = \tfrac{3}{20}$$
$$+ P(J_1)P(RWR | J_1) \rightarrow \left(\tfrac{3}{6}\right)\left(\tfrac{3}{5}\right)\left(\tfrac{2}{4}\right) = \tfrac{3}{20}$$
$$+ P(J_1)P(WRR | J_1) \rightarrow \left(\tfrac{3}{6}\right)\left(\tfrac{3}{5}\right)\left(\tfrac{2}{4}\right) = \tfrac{3}{20}$$

${}_3C_2$

$$= P(J_1)\left(\tfrac{3}{20} + \tfrac{3}{20} + \tfrac{3}{20}\right)$$
$$= 3\,P(J_1)\left(\tfrac{3}{20}\right)$$
$$= 3\left(\tfrac{1}{2}\right)\left(\tfrac{3}{20}\right) = \tfrac{9}{40}$$

Considering J_2, $P(J_2)P(R_2W | J_2) = {}_3C_2\, P(J_2) \overset{R}{\left(\tfrac{4}{6}\right)} \overset{R}{\left(\tfrac{3}{5}\right)} \overset{W}{\left(\tfrac{2}{4}\right)}$

$$= 3\left(\tfrac{1}{2}\right)\left(\tfrac{1}{5}\right) = \tfrac{3}{10}$$

Solution: $P(J_2 | R_2W) = \dfrac{3/10}{9/40 + 3/10} = \boxed{\dfrac{4}{7}}$