Bootstrapping

Nate Wells

Math 141, 3/12/21

Outline

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- Review how the sampling distribution can be used to assess sampling variability
- Discuss bootstrapping as means of approximating the sampling distribution

Section 1

Sampling Distribution

Polling Example

- A Oct 29 - Nov 1 2020 poll by Marist College surveyed 1020 registered voters in Pennsylvania by landline or mobile number, asking

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  - The survey lists a margin of error of $\pm 3.8\%$, with 95% confidence (we'll discuss this later)
- In the Nov. 3 2020 election, Biden/Harris had 50.01% of the vote, while Trump/Pence had 48.84% of the vote.

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- Is it **biased**? Yes. Although hopefully bias was reduced through use of survey weighting.

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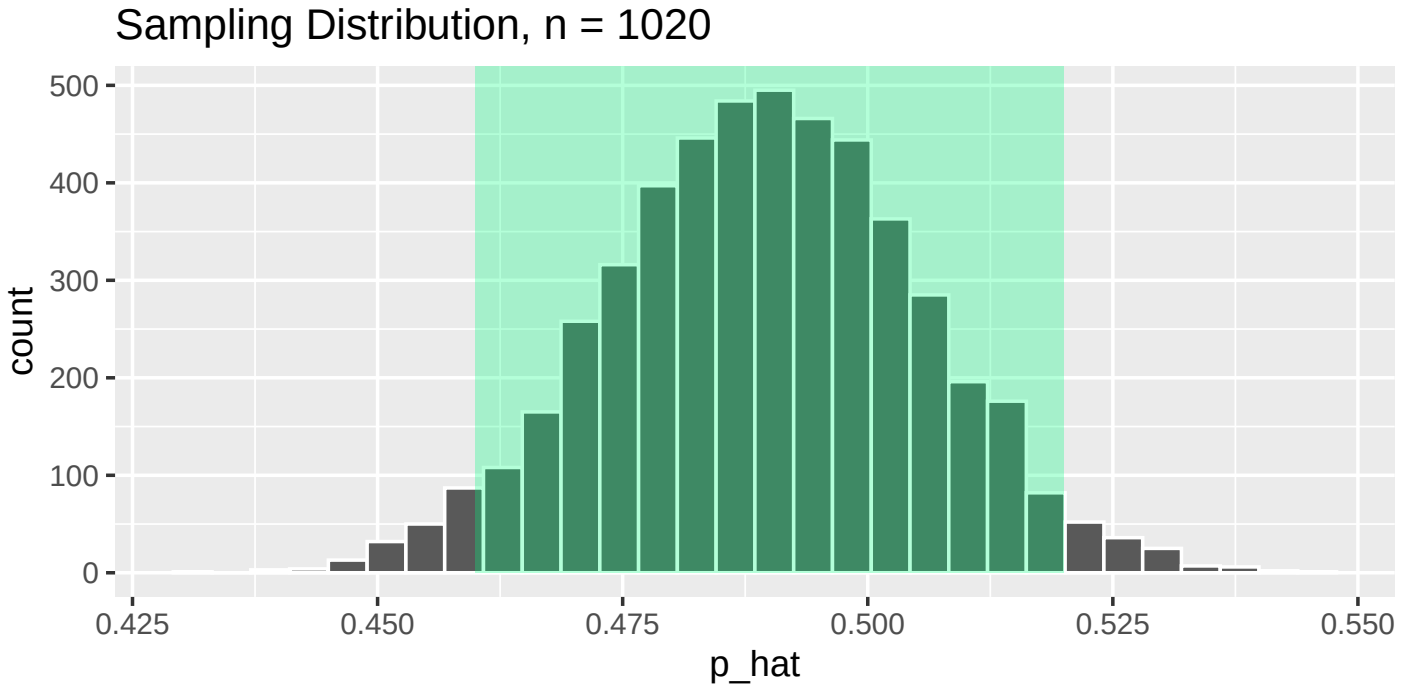

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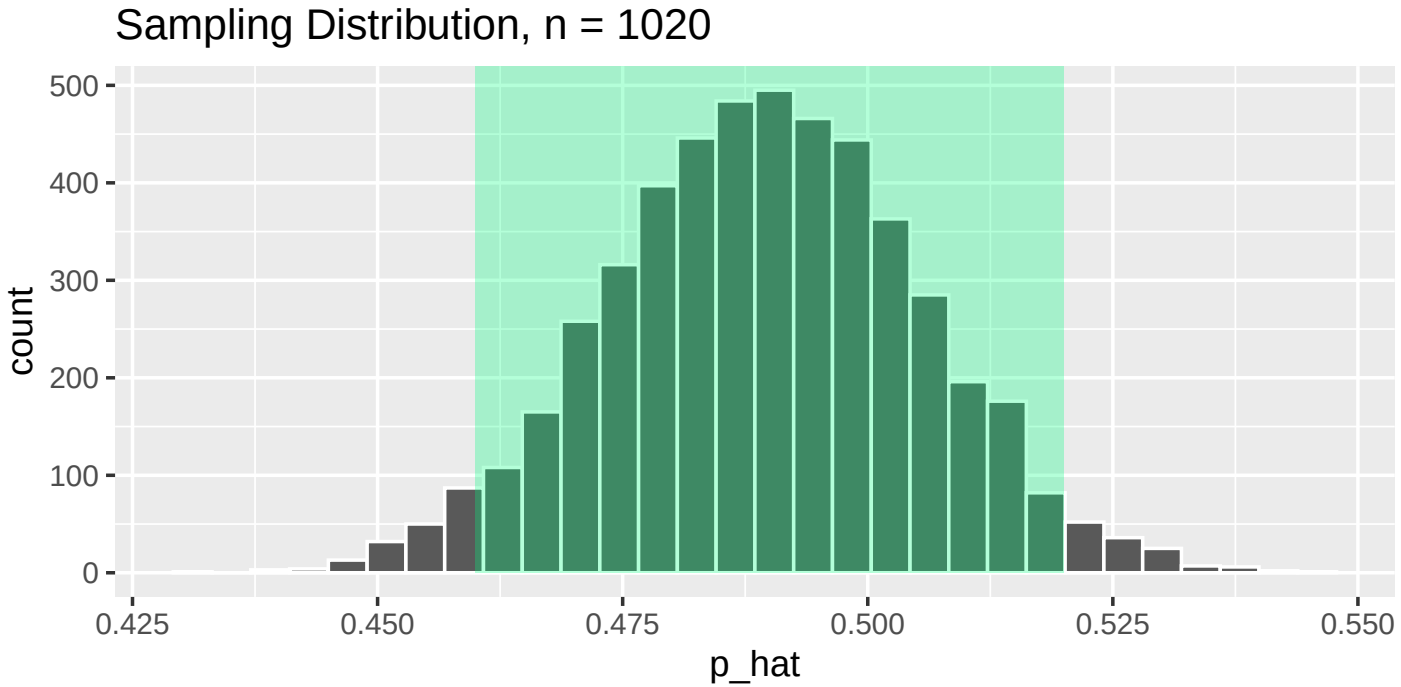

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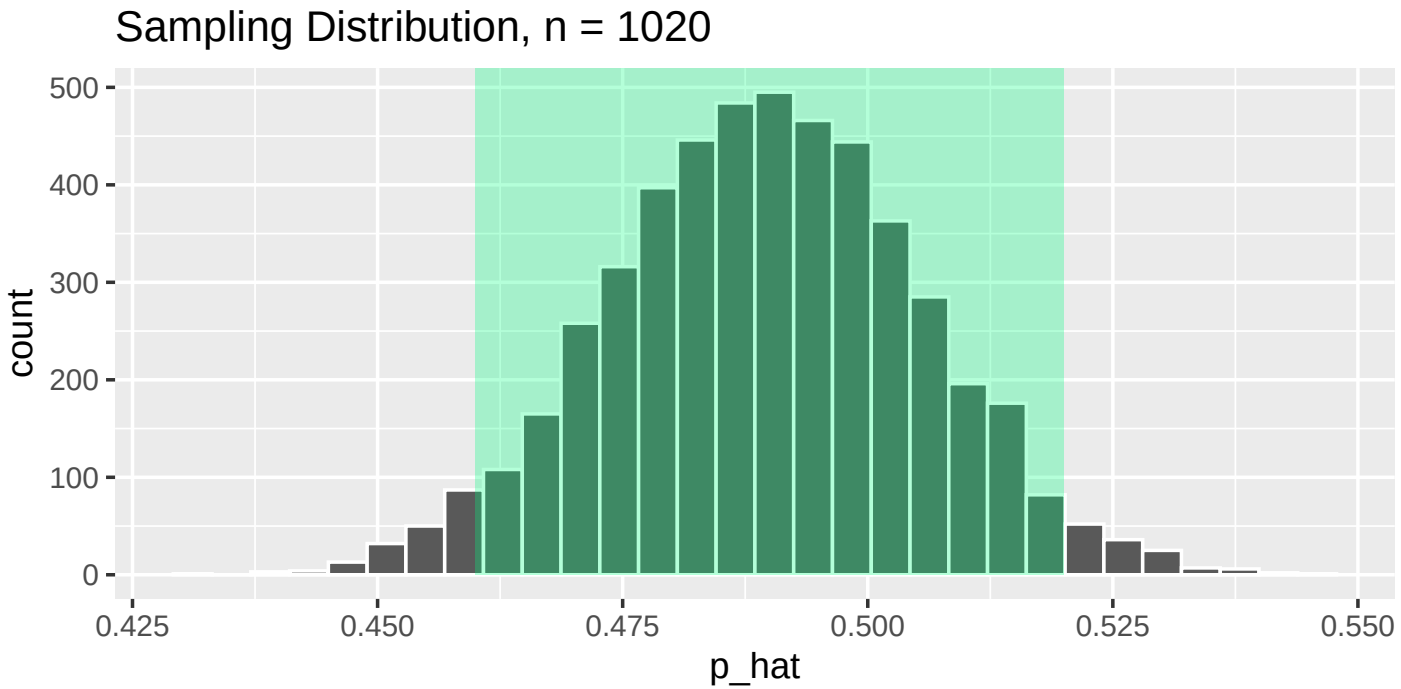

- Of these, only 254 (5%) differed from the true value $p = .49$ by more than .03
- But this also means that for 95% of samples, the true proportion p is within 0.03 of the sample proportion $\hat{p}$.

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Section 2

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  - The phrase originated in the 19th century as reference to a ludicrous or impossible feat
  - By the mid 20th century, its meaning had changed to suggest a success by one's own efforts, without outside help
- Its use in statistics alludes to both interpretations.

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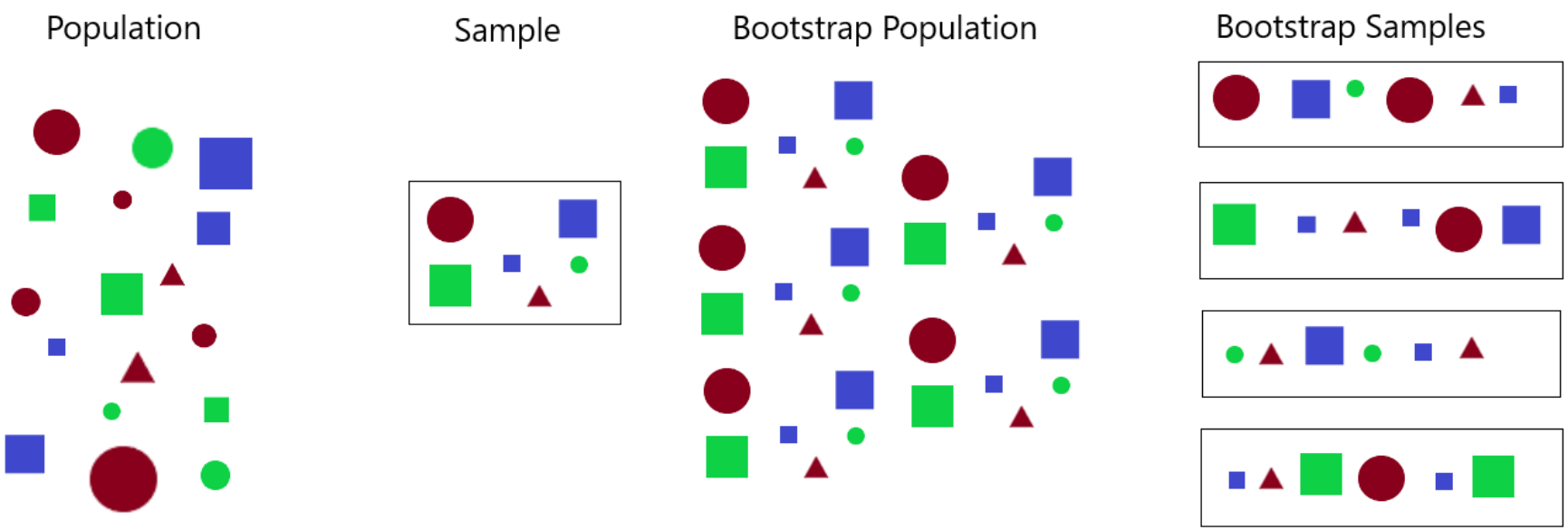
- Resampling from the sample approximates sampling many times from the population
- The distribution of statistics from the resamples approximates the sampling distribution

Theory


Population
Sample
Bootstrap Population
Bootstrap Samples

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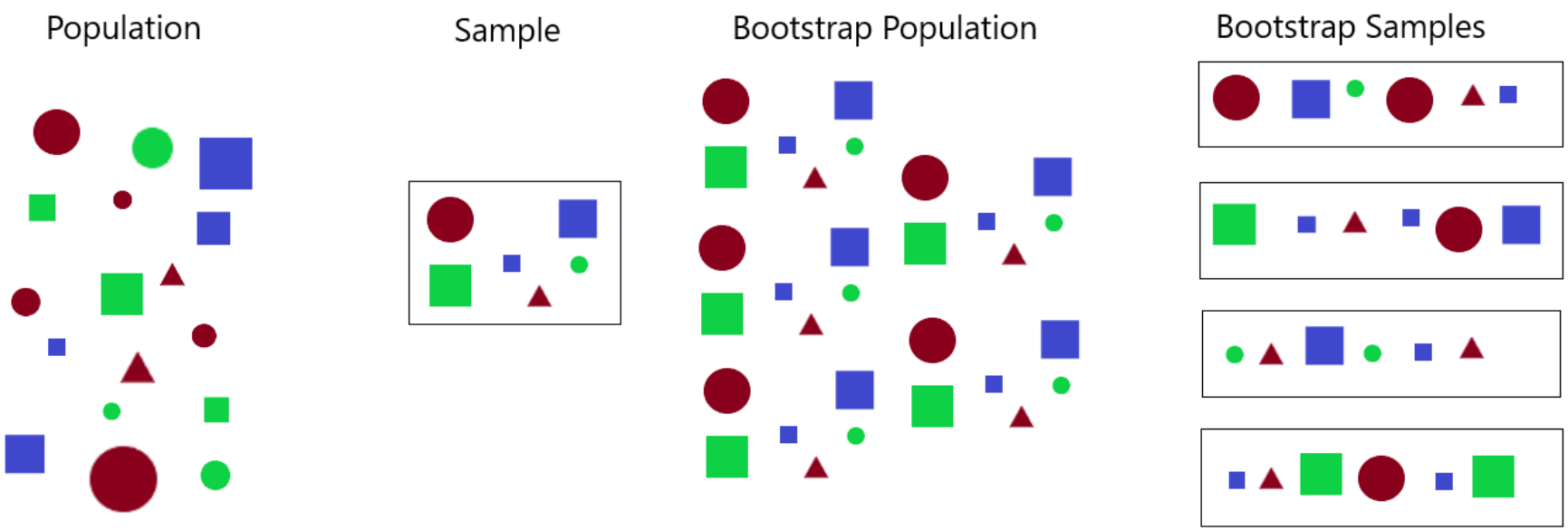

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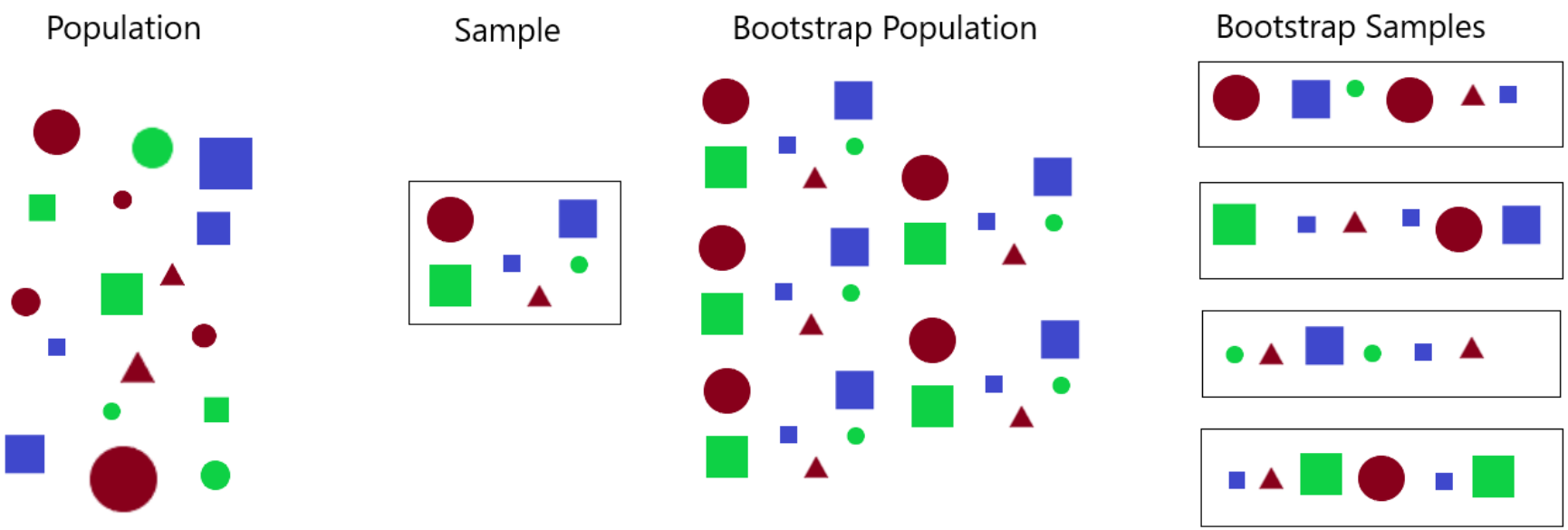

- We could copy the original sample many times to create a bootstrap population, and then sample without replacement to get bootstrap samples
- But this is the same as sampling **with** replacement from the original sample

The Bootstrap Procedure

To generate a **bootstrap distribution**:

1. Obtain an SRS of size n from the population.
2. Generate a bootstrap sample of size n by resampling *with* replacement from the original sample
3. Repeat (2) a large number of times (with technology, at least 1000 times)
4. For each bootstrap sample, calculate the appropriate statistic (called the **bootstrap statistic**)
5. The collection of all bootstrap statistics form the **bootstrap distribution**

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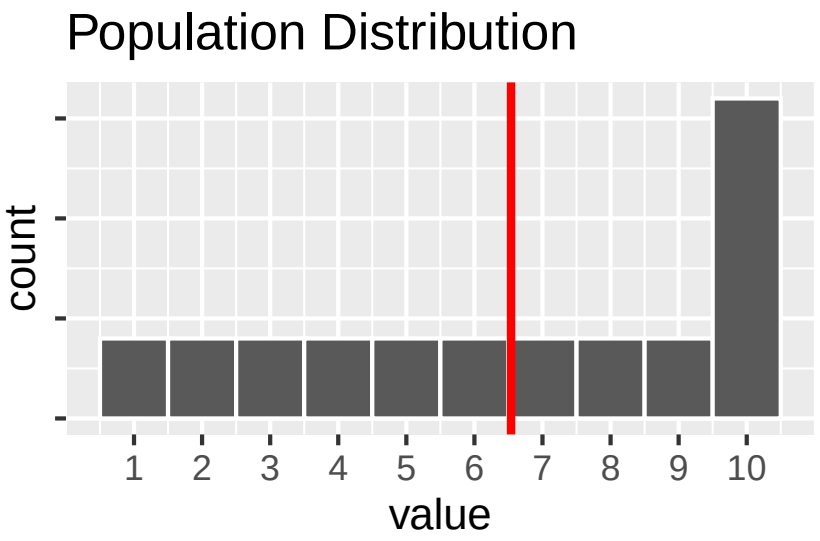
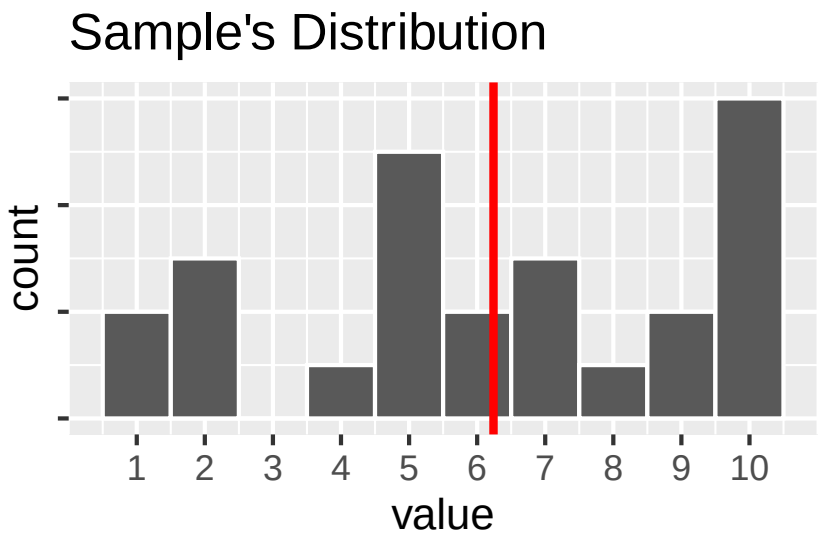
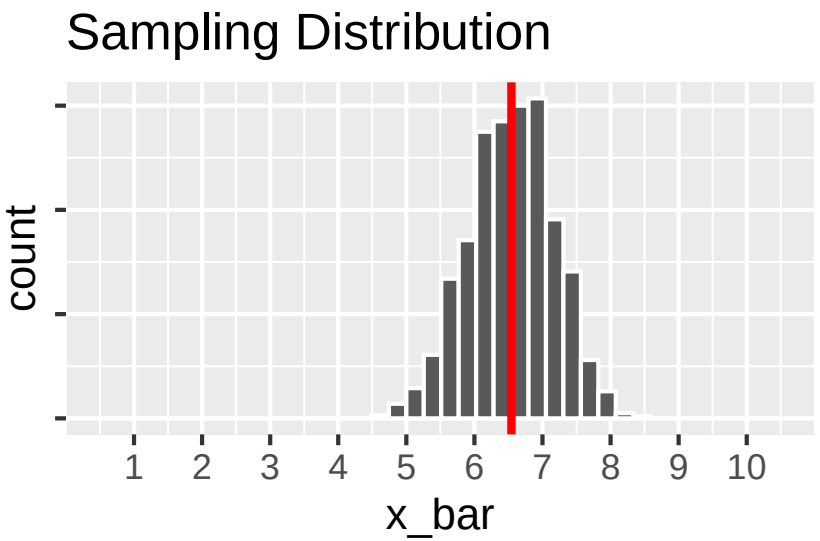
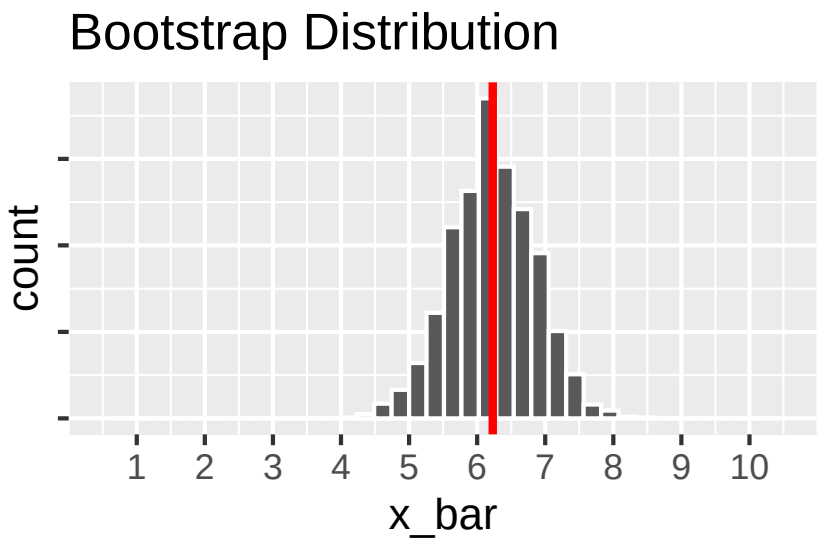
- Suppose we draw a sample hand of size 25 and calculate the mean value of the hand.
- Since we have the deck of cards, we can look at:
  1. The population distribution
  2. The single sample's distribution
  3. The sampling distribution for sample means
  4. The bootstrap distribution for sample means

House of Cards









House of Cards

We can compute some relevant statistics:

Population:

mean_value	sd_value
6.538462	3.153211

Sample:

mean_value	sd_value
6.24	3.072458

Sampling Distribution:

mean_xbar	sd_xbar
6.54342	0.6291307

Bootstrap Distribution:

mean_xbar	sd_xbar
6.2289	0.6240862

Reproduction Rate for Covid-19

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## 4        3 12
## 5        4  5
## 6        6  1
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- Is the true reproduction rate exactly 2.06?
  - Surely not! This is just one sample of size 50
- But how much does the reproduction rate vary from sample to sample?

Bootstrap Reproduction Rate

Create the bootstrap samples:

```
bootstrap_samples <- covid %>%
  rep_sample_n(size = 50, replace = TRUE, reps = 2000)

head(bootstrap_samples)
```

```
## # A tibble: 6 x 2
## # Groups:   replicate [1]
##   replicate infected
##       <int>    <int>
## 1         1        2
## 2         1        1
## 3         1        1
## 4         1        0
## 5         1        1
## 6         1        0
```

Bootstrap Reproduction Rate

Compute bootstrap statistics:

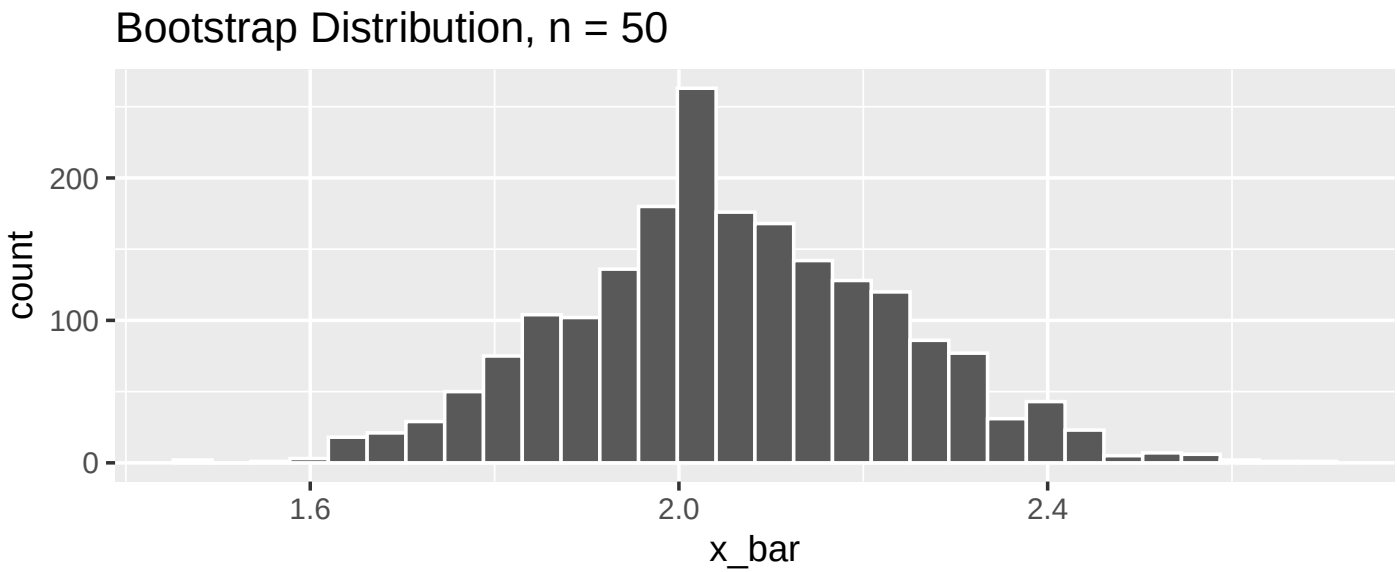
```
bootstrap_stats <- bootstrap_samples %>%
  group_by(replicate) %>%
  summarize(x_bar = mean(infected))

head(bootstrap_stats)
```

```
## # A tibble: 6 x 2
##   replicate x_bar
##       <int> <dbl>
## 1         1  1.86
## 2         2  2.36
## 3         3  2.22
## 4         4  1.86
## 5         5  1.88
## 6         6  1.6
```

Bootstrap Reproduction Rate

Graph the bootstrap distribution:

```
ggplot(bootstrap_stats, aes(x = x_bar))+
  geom_histogram(bins = 30, color = "white")+
  labs(title = "Bootstrap Distribution, n = 50")
```



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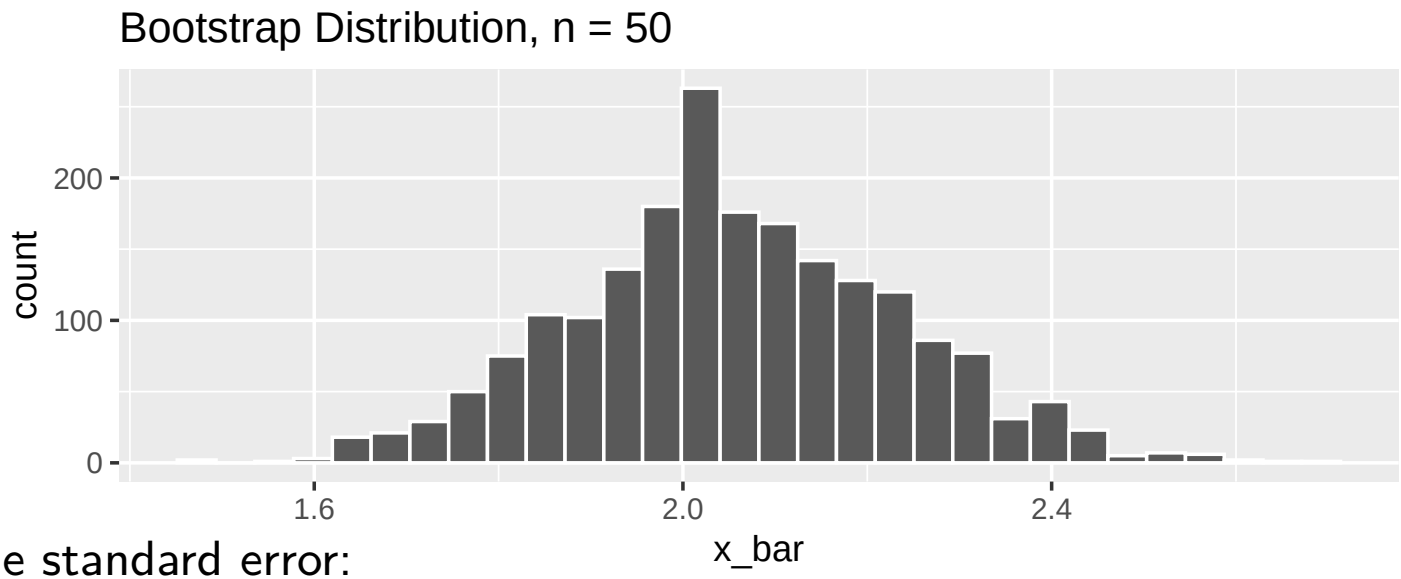

Estimate the standard error:

```
bootstrap_stats %>% summarize(SE = sd(x_bar))
```

```
## # A tibble: 1 x 1
##       SE
##    <dbl>
## 1 0.177
```