Confidence Intervals

Nate Wells

Math 141, 3/15/21

Outline

In this lecture, we will...

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- Introduce confidence intervals as a method for estimating a parameter
- Use bootstrapping as means of creating confidence intervals
- Implement the infer package to calculate confidence intervals

Section 1

Confidence Intervals

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- It might be preferable to estimate the proportion using a range of values, with smaller intervals corresponding to larger samples.
 - With just n = 10 people, you might give a range 0.2 to 0.8 for p.
 - But with n = 100, you might instead give the range 0.4 to 0.6.

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• To get the margin of error and the confidence level, we make use of the sampling distribution (or the bootstrap approximation).

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 - How do we get the SE? Bootstrap.

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##	3	2	14
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##	6	6	1
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infected ## n ## 1 0 5 1 13 ## 2 ## 3 2 14 3 12 ## 4 4 5 ## 5 ## 6 6 1 ## mean infected ## 1 2.06

- Is the true reproduction rate exactly 2.06?
 - Surely not! This is just one sample of size 50
- But how much does the reproduction rate vary from sample to sample?

```
Create the bootstrap samples:
bootstrap_samples <- covid %>%
    rep_sample_n(size = 50, replace = TRUE, reps = 2000)
```

```
head(bootstrap_samples)
```

```
## # A tibble: 6 x 2
## # Groups: replicate [1]
     replicate infected
##
##
         <int>
                   <int>
## 1
                        2
              1
## 2
              1
                        1
## 3
              1
                        1
              1
                        0
## 4
## 5
              1
                        1
              1
                        0
## 6
```

```
Compute bootstrap statistics:
bootstrap_stats <- bootstrap_samples %>%
group_by(replicate) %>%
summarize(x_bar = mean(infected))
head(bootstrap_stats)
```

```
## # A tibble: 6 x 2
##
     replicate x_bar
##
         <int> <dbl>
## 1
             1 1.86
## 2
             2 2.36
             3 2.22
## 3
            4 1.86
## 4
            5 1.88
## 5
## 6
            6 1.6
```

```
Graph the bootstrap distribution:
ggplot(bootstrap_stats, aes(x = x_bar))+
  geom_histogram(bins = 30, color = "white")+
  labs(title = "Bootstrap Distribution, n = 50")
```



Bootstrap Distribution, n = 50

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```
## # A tibble: 1 x 1
## SE
## <dbl>
## 1 0.177
```

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- Our best guess for the reproduction rate is between 1.705 and 2.415. This method has a success rate of 95%.
- For reference, this interval matches the one provided by the WHO on 1/23/20.

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General Confidence Intervals

The C% confidence interval for a parameter is an interval estimate that is computed from sample data by a method that captures the parameter for C% of all samples.

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 - Then 80% of sample means are between L and U
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 - Then 80% of sample means are between L and U
 - L is called the 10th percentile and U is called the 90th percentile
- If we build an interval around each sample mean \bar{x} of length U L, then our interval will capture the true parameter for 80% of all samples

- In practice, we won't be able to look at the sampling distribution to find L and U.
 - Instead, we find the corresponding percentiles from the bootstrap distribution.



Bootstrap Distribution

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Bootstrap Distribution

```
    We can use the quantile function in R to calculate L and U:
my_boot %>% summarise(L = quantile(x_bar, .1),
U = quantile(x_bar, .9))
```

```
## # A tibble: 1 x 2
## L U
## <dbl> <dbl>
## 1 6.83 11.7
```

Section 2

The infer package

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- To investigate the infection rate

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covid %>%
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- The resulting data frame has a number of rows equal reps \times sample_size

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• After applying calculate the resulting data frame consists of one bootstrap statistic for each replicate (saved to the variable stat)

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covid_boot<- covid %>%
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  calculate(stat = "mean")
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Simulation-Based Bootstrap Distribution

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percentile_ci<-covid_boot %>%
  get_ci(level = .95, type = "percentile")
percentile_ci
```

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## lower_ci upper_ci
## <dbl> <dbl>
## 1 1.7 2.42
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- When using the percentile type, the first value printed is the lower and the second is the upper bound.
 - The headings indicate that these are the corresponding percentiles in the bootstrap distribution

Shade Confidence Intervals

• Once you've used get_ci to obtain endpoints of the confidence interval, you can shade the sampling distribution with the confidence interval region.

```
covid_boot %>% visualize()+shade_ci(endpoints = percentile_ci)
```



Simulation-Based Bootstrap Distribution

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```