Probability

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Math 141, 3/26/21

Outline

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- Introduce key definitions for probability theory
- Discuss conditional probability

Section 1

Probability Theory

Introduction

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- Probability has diverse applications:
  - *Statistics*: Quantifies uncertainty for random sampling
  - *Physics*: Explains the universe on its smallest scale via quantum mechanics
  - *Biology*: Predicts results of population dynamics and ecosystems
  - *Computer Science*: Implements random algorithms in machine learning for problem solving
  - *Economics*: Models volatile market trends and interactions of financial systems
  - *Philosophy*: Provides framework for understanding knowledge and certainty in epistemology

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  - An outcome or event is something you could observe as the result of a random experiment.
  - Suppose a 6-sided die is rolled. Both "the die rolled a 1" and "the die rolled a 1 or a 2" are events.

The Law of Large Numbers

Theorem (The Law of Large Numbers)

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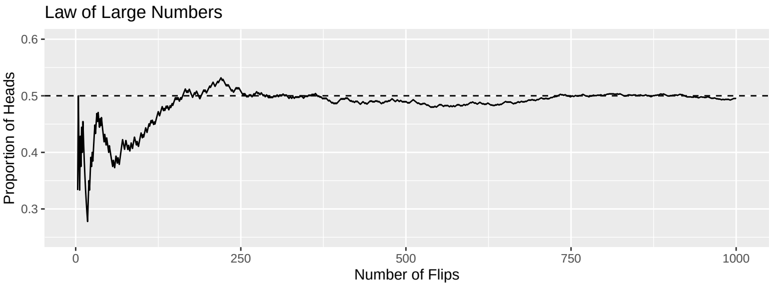

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$$P(\text{ starts with f }) = P(\text{ roll a 4 or roll a 5 }) = P(\text{ roll a 4 }) + P(\text{ roll a 5 }) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

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  - What is a certain outcome for this experiment?

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$$P(\text{ roll something other than a 1 }) = 1 - P(\text{ roll a 1}) = 1 - \frac{1}{6} = \frac{5}{6}$$

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Theorem (Multiplication Rule)

If A and B are independent events, then the probability that both occur is the product of their individual probabilities:

$$P(A \text{ and } B) = P(A)P(B)$$

- Suppose I want to choose a sample of people in our 141 classes. Assume that I do so 1 at a time, with replacement. What is the probability that when I choose 3 times, nobody in the sample is from the 10am section? (30 / 56 students are in the 10am section)

Section 2

Conditional Probability

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4. What is the probability that a random sophomore preferred coffee?

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- Find the probability that a student is a sophomore who likes coffee.

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- In the previous example, are the events that "a student is a sophomore" and "a student prefers coffee" independent?

The Law of Total Probability

One useful trick for computing probabilities is the following:

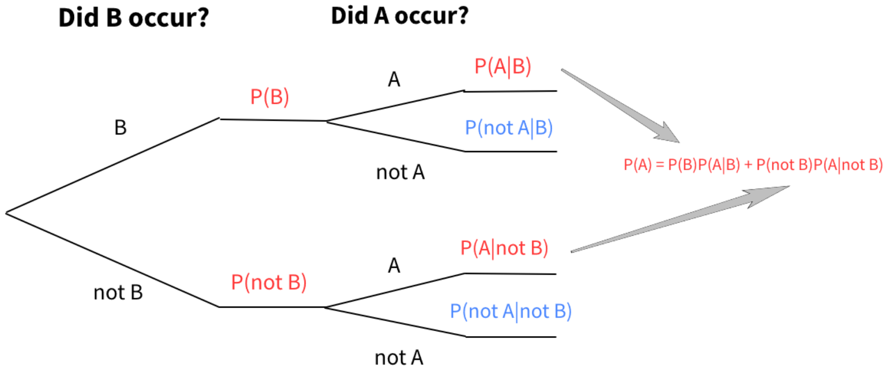
Theorem (The Law of Total Probability)

Let A and B be events. Then

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

- We can often represent the Law of Total Probability using a Tree Diagram:

Tree Diagrams



Lost Marbles

Two boxes contain a different number of red and blue marbles. The first box contains 20% red marbles while the second contains 80% red marbles. Suppose we select a marble from box 1 25% of the time and a marble from box 2 75% of the time. **What is the probability that a red marble is selected?**