Probability

Nate Wells

Math 141, 3/26/21

Outline

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- Discuss the Law of Total Probability and Bayes' Rule
- Define and investigate Random Variables

Section 1

Conditional Probability

The Law of Total Probability

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One useful trick for computing probabilities is the following:

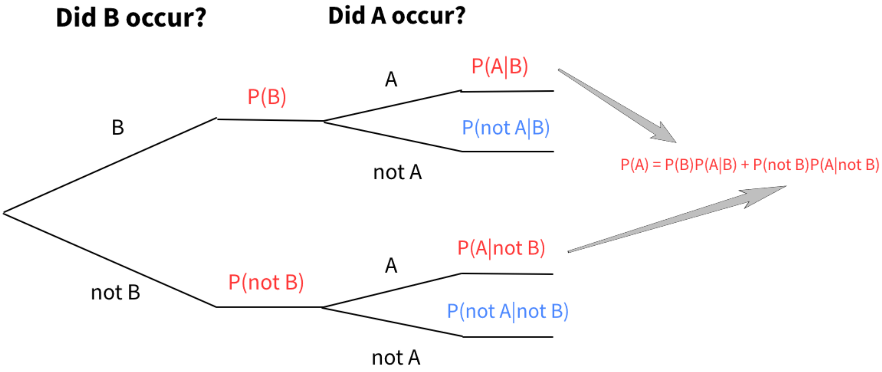
Theorem (The Law of Total Probability)

Let A and B be events. Then

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

- We can often represent the Law of Total Probability using a Tree Diagram:

Tree Diagrams



Lost Marbles

Two boxes contain a different number of red and blue marbles. The first box contains 20% red marbles while the second contains 80% red marbles. Suppose we select a marble from box 1 25% of the time and a marble from box 2 75% of the time. **What is the probability that a red marble is selected?**

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  - What is $P(A|B)$ in this case?

Bayes' Rule and Hypothesis Testing

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- Express each of these in terms of the p-value and conditional probabilities.

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  - $P(H)$ is very, very low compared to $P(O)$.
- By the principle of hypothesis testing, we might see that $P(O|H)$ is high and favor the monster hypothesis H.
- Hypothesis testing just tells us about consistency of data with the null hypothesis. It doesn't give us the probability that the null is true.

Section 2

Random Variables

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- We use equation to express events associated to random variables.
  - Let $X = 5$ denotes the event "The random variable X takes the value 5".
- Events associated to variables have probabilities of occurring.
  - $P(X = 5) = .5$ means X has 50% probability of taking the value 5.

Types of Random Variables

There are two main types of random variables:

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  - The amount of time it takes a radioactive particle to decay.
- Some discrete variables can be well-described by continuous variables:
  - The height of a random person selected from a large population.
  - The proportion of heads in a long sequence of coin flips.

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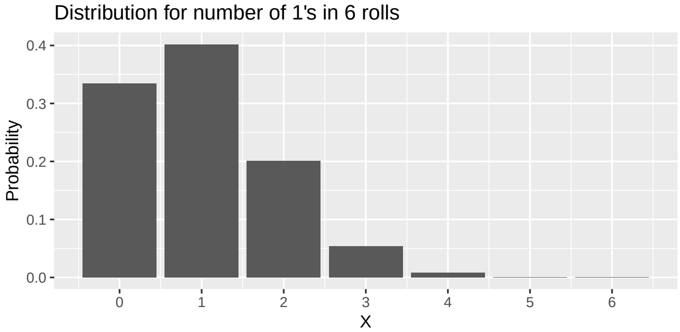

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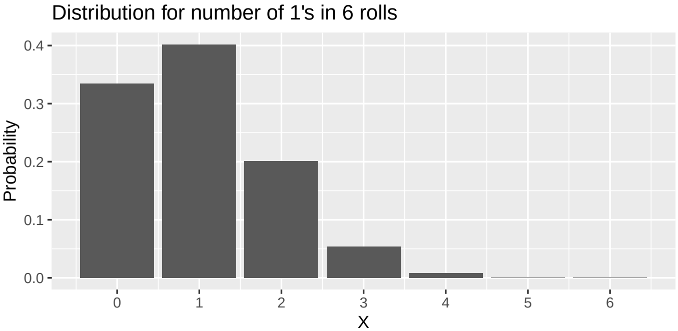

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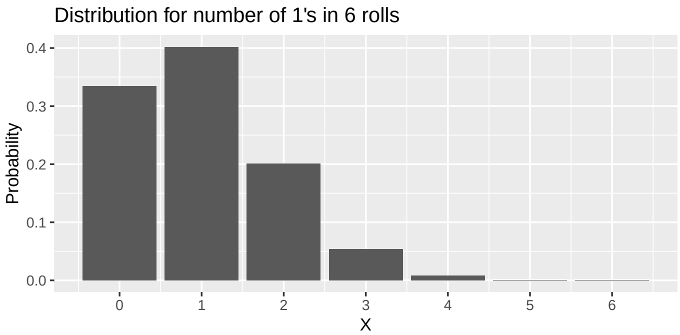

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  - Calculate $P(X \leq 1)$. Then find x so that $P(X \leq x) \geq .75$.

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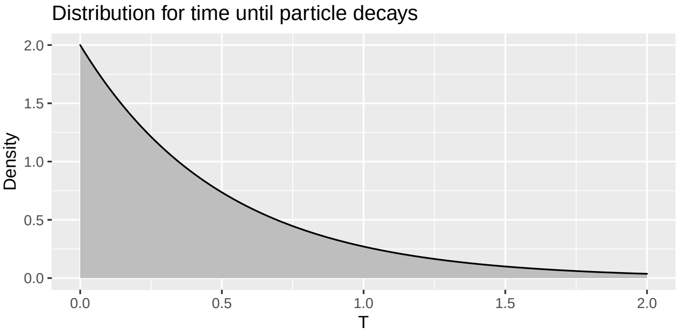

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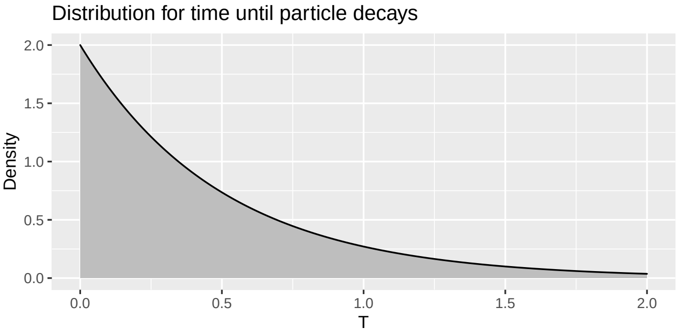

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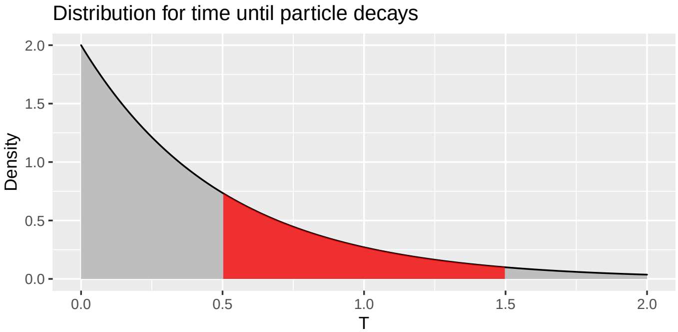

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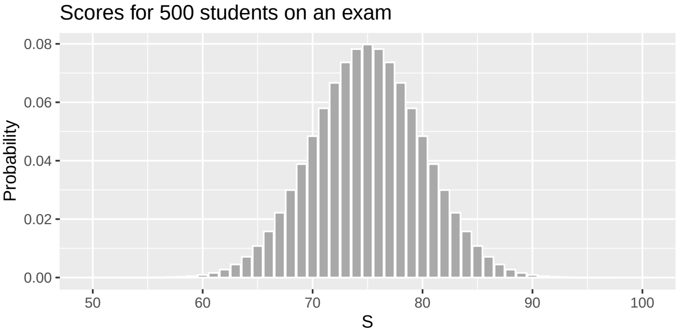

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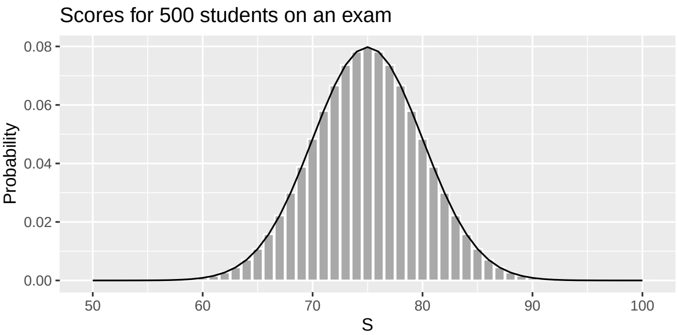

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$$\begin{aligned} E[X] =& 1P(X = 1) + 2P(X = 2) + 3P(X = 3) + 4P(X = 4) + 5P(X = 5) \\ =& 1\frac{2}{10} + 2\frac{4}{10} + 3\frac{1}{10} + 4\frac{1}{10} + 5\frac{2}{10} = \frac{27}{10} = 2.7 \end{aligned}$$

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- But also notice that

$$\begin{aligned} E[X] =& \frac{1}{10}\left(1 \cdot 2 + 2 \cdot 4 + 3 \cdot 1 + 4 \cdot 1 + 5 \cdot 2\right) \\ =& \frac{1}{10}\left(1 + 1 + 2 + 2 + 2 + 2 + 3 + 4 + 5 + 5\right) \end{aligned}$$

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This is a generalization:

Theorem (The Law of Large Numbers)

Let X be a random variable whose value depends on a random experiment. Suppose the experiment is repeated n times and let $\bar{x}_n$ denote the arithmetic mean of the values of X in each trial. As n gets larger, the arithmetic mean $\bar{x}_n$ approaches the expected value $E[X]$ of that variable.

Gambler's Ruin

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- Assuming each wedge has equal probability, what is the expected value of the bet?

- Suppose a gambler begins with $10,000. What will the gambler's fortune look like after 1000 plays?