Inference for Many Means

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Math 141, 4/23/21

Outline

In this lecture, we will...

- Construct a statistic to measure the differences in mean among several groups
- Discuss the theoretical and simulation-based distribution of the F statistic
- Use ANOVA to test for a difference in means among several groups

Section 1

Differences Among Several Populations

Comparing a Quantatiative and Categorical Variable

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  - Used the chi-square test to determine whether two categorical variables were independent
  - Determine whether a quantitative variable and categorical variable (with only 2 levels) were independent
- The Analysis of Variance (ANOVA) test will allow us to assess whether the mean values of a quantitative variable differ across the levels of a categorical variable.

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Research Question: Certainly, individual tastes in movie genres vary. But in aggregate, do audience ratings of movies depend on genre? To answer, we assess the Rotten Tomatoes audience rating for 132 films from 2011 spread across 7 genress.

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- **Observational unit**: a single film
- **Sample**: 132 films from 2011
- **Population**: All films (maybe from last 20 years?)
- **Variables**: Audience Rating and Genre
- **Parameters**: Average audience rating for each genre, $\mu_1, \ldots, \mu_7$.
- **Null Hypothesis**: $H_0 : \mu_1 = \mu_2 = \cdots = \mu_7$
- **Alternative Hypothesis**: At least one μ is not equal to the others

Data Exploration

Do ratings differ by genre?

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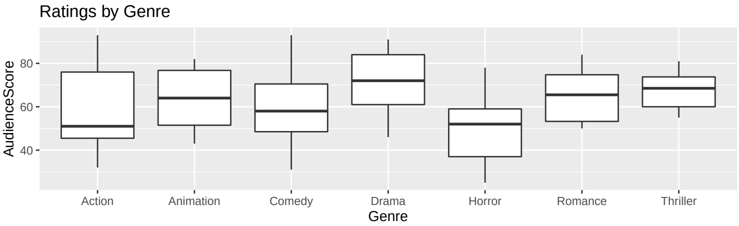

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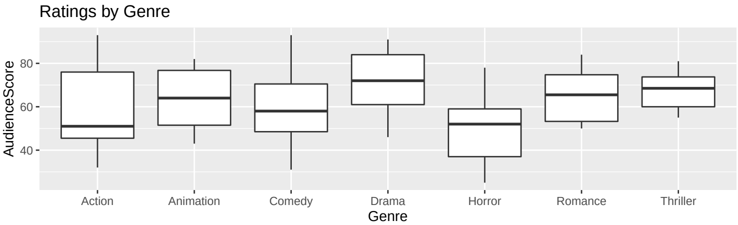


```
movies %>% group_by(Genre) %>%
  summarize(number = n(), avg_rating = mean(AudienceScore), st_dev = sd(AudienceScore))
```

```
## # A tibble: 7 x 4
##   Genre     number avg_rating st_dev
##   <fct>      <int>      <dbl>  <dbl>
## 1 Action        32       58.6  18.4
## 2 Animation     12       64.1  13.9
## 3 Comedy        27       59.1  15.7
## 4 Drama         21       72.1  14.5
## 5 Horror        17       48.6  15.9
## 6 Romance       10       64.8  12.9
## 7 Thriller      12       67.7   9.01
```

```
##       Movie number avg_rating st_dev
## 1 All Films    131         61     17
```

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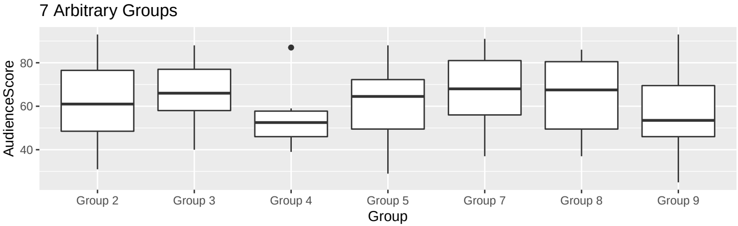

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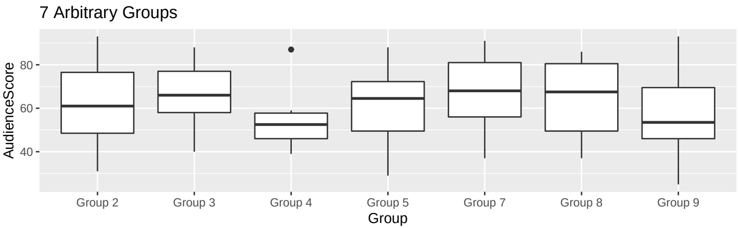

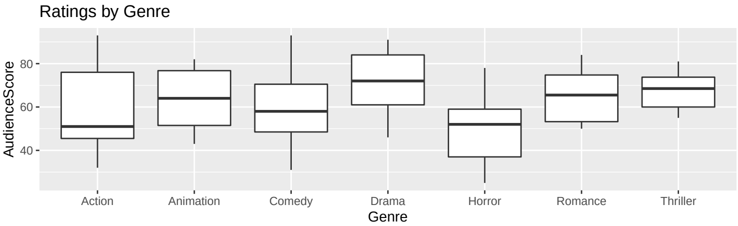

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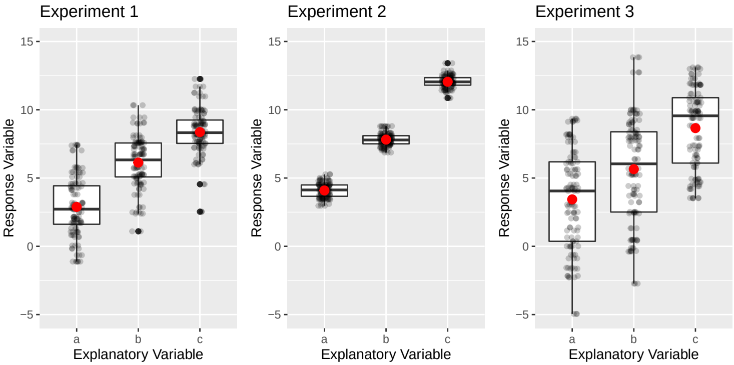

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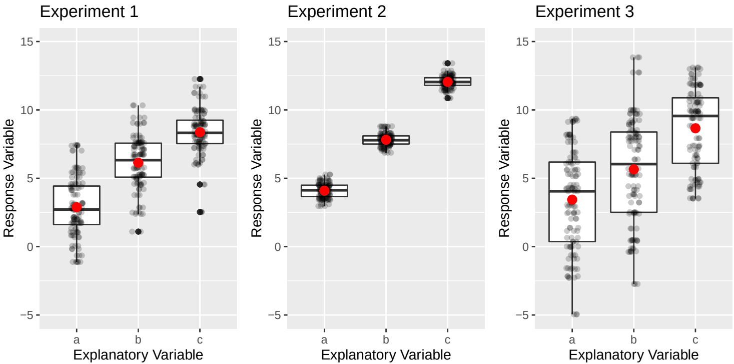

- **Strongest**: Experiment 2
- **Weakest**: Experiment 3

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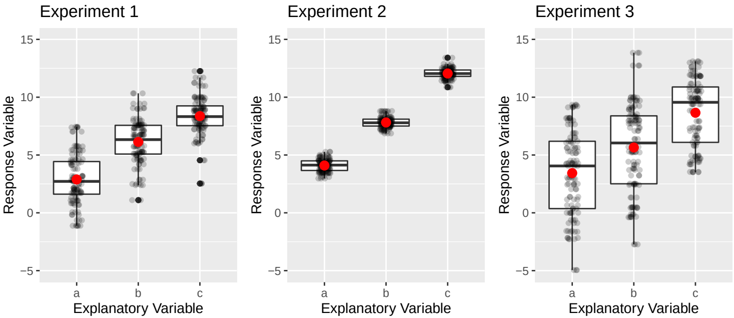
- Is the variation observed among sample means greater than what can be explained by variability in observations within each group alone?

Partitioning Variability



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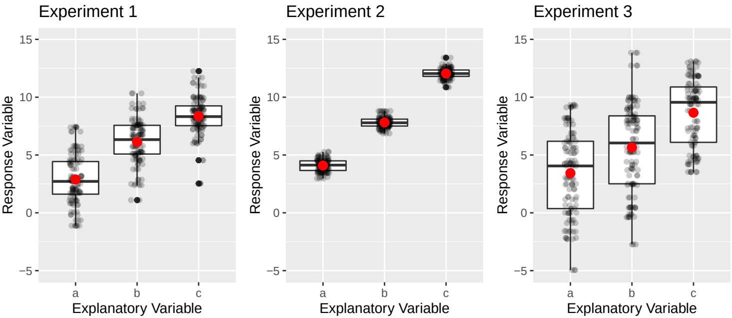

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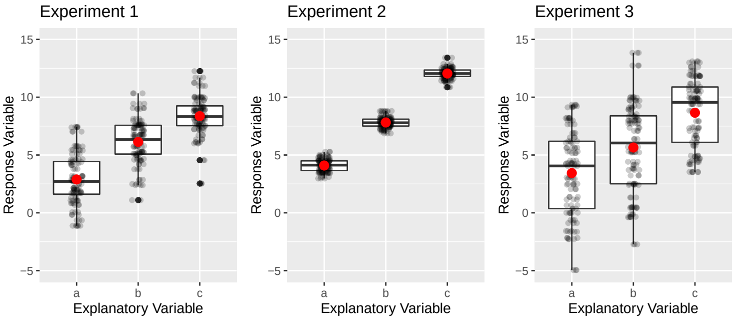

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- Variability Within Groups: How much do observations in groups vary from mean?
  - Within each group, compare black dots to red dot

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$$\begin{aligned}\text{Variability Within Groups } &= \sum (x - \bar{x})^2\\ &= \text{Sum of Squares Total}\\ &= \text{SST}\end{aligned}$$

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- Our goal is to use MSG and MSE to build a test statistic which measures when variability between groups is much greater than variability within groups

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$$F > 1$$

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movies_F<- movies %>%
  specify(AudienceScore ~ Genre) %>%
  calculate(stat = "F")
movies_F
```

```
## # A tibble: 1 x 1
##    stat
##   <dbl>
## 1  4.34
```

- Is this a large value of F?

Section 2

The Distribution of the F statistic

The setup for Hypothesis Tests

- Hypotheses
  - **Null Hypothesis**: $H_0 : \mu_1 = \mu_2 = \cdots = \mu_7$
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- We can find the distribution F under the null hypothesis by...
  - Randomization
  - Theoretical Approximation.

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- We can imitate drawing new samples from this population by `permuting` the group labels among observations
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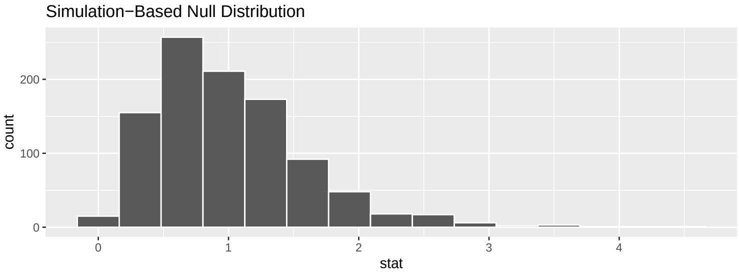
- This way, we can study how the size of the F statistic changes just due to random sampling

Randomization and Permutation II

```
null_dist<-movies %>%
  specify(AudienceScore ~ Genre) %>%
  hypothesize(null = "independence") %>%
  generate(reps = 1000, type = "permute" ) %>%
  calculate(stat = "F")
null_dist %>% visualize()
```



- Most F statistics are at most 3
  - i.e. Assuming independence, Variance BETWEEN groups is at most 3 times variance WITHIN groups

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null_dist %>% visualize()+shade_p_value(obs_stat = movies_F, direction = "right")
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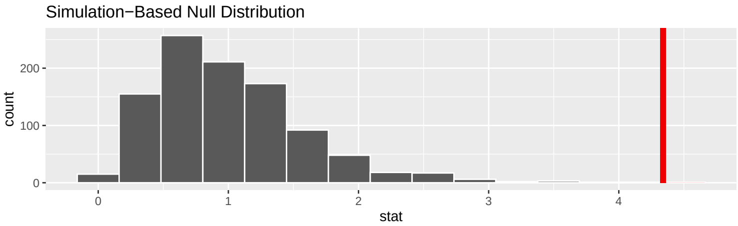

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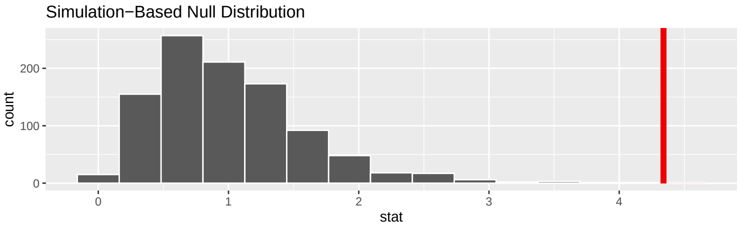


```
null_dist %>% get_p_value(obs_stat = movies_F, direction = "right")
```

```
##   p_value
## 1   0.001
```

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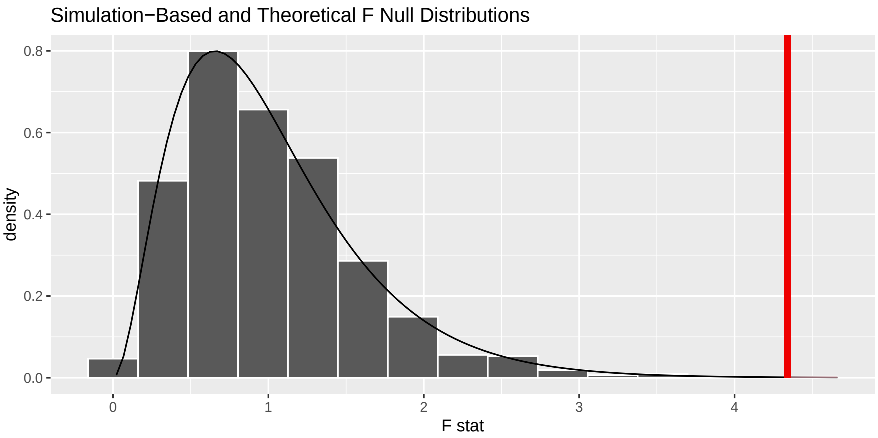
  - The p-value is the area in the right tail.

```
p_value<- pf(q = 4.340672, df1 = 6, df = 125, lower.tail = FALSE)
p_value
```

```
## [1] 0.00052
```

Theory-based and Simulation-based Distributions



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There is No Accounting for Taste . . . Even on Average

- The observed F statistic had P-value less than $\alpha = 0.001$
    - This gives extremely good evidence against the Null hypothesis.
    - We conclude that Audience Rating does depend on genre.