Probability

Nate Wells

Math 141, 3/30/22

Outline

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- Discuss the Law of Total Probability and Bayes' Rule
- Define and investigate Random Variables

Section 1

Conditional Probability

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- Suppose we draw two cards (without replacement) from a standard 52 card deck. What is the probability that both cards are aces?

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- In the previous example, are the events that "1st card is an Ace" and "2nd card is an Ace" independent?

Multiplication Rule for Independent Events

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  - Since the coin presumably gives us no information about the die, we say A and B are independent.
  - What is the probability that A and B both occur?

The Law of Total Probability

One useful trick for computing probabilities is the following:

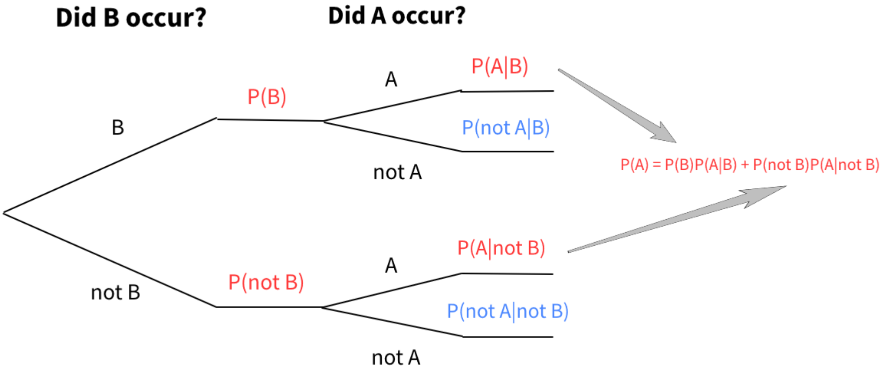
Theorem (The Law of Total Probability)

Let A and B be events. Then

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

- We can often represent the Law of Total Probability using a Tree Diagram:

Tree Diagrams



Lost Marbles

Two boxes contain a different number of red and blue marbles. The first box contains 20% red marbles while the second contains 80% red marbles. Suppose we select a marble from box 1 25% of the time and a marble from box 2 75% of the time. **What is the probability that a red marble is selected?**

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To relate $P(A|B)$ and $P(B|A)$, we use the following theorem:

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  - What does this suggest about A and B?
  - What is $P(A|B)$ in this case?

Testing for a Rare Disease

Suppose that we have a rapid COVID-19 test that correctly diagnoses a person who **does not** have COVID 99% of the time, and correctly diagnoses a patient who **does** have COVID 80% of the time. Suppose a person takes this test and receives a positive diagnosis. Assume that the overall prevalence of COVID at the time of the test was 0.5%. **What is the probability that the person has COVID**?

Section 2

Random Variables

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- We use equations to express events associated to random variables.
  - I.e "$X = 5$" represents the event "The random variable X takes the value 5".
- Events associated to variables have probabilities of occurring.
  - $P(X = 5) = .5$ means X has 50% probability of taking the value 5.

Types of Random Variables

There are two main types of random variables:

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  - The temperature of my office at a particular time of the day.
  - The amount of time it takes a radioactive particle to decay.
- Some discrete variables can be well-described by continuous variables:
  - The height of a random person selected from a large population.
  - The proportion of heads in a long sequence of coin flips.

The Distribution of a Random Variable

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    - the values the variable takes, and the *frequency* the variable takes those values.

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- Suppose instead that I have a purse filled with the following 100 coins:

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- Playing the casino game is very similar to drawing a random coin from the purse.

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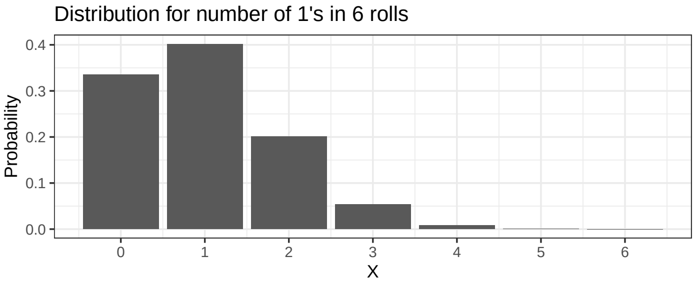

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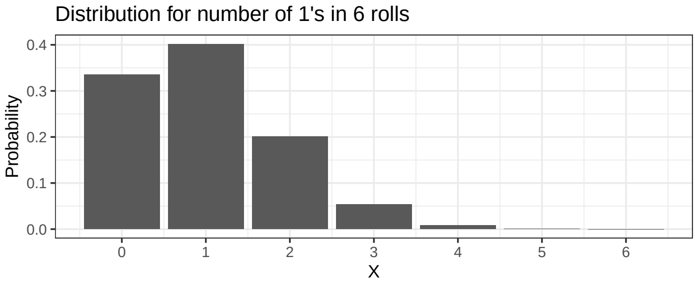

- Heights of bars are probabilities
  - This is analogous to rescaling a histogram to have heights equal to proportions, rather than counts

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The **expected value** (or mean) of a discrete random variable X is

$$E[X] = x_1 P(X = x_1) + x_2 P(X = x_2) + \dots x_n P(X = x_n) = \sum_{i=1}^{n} x_i P(X = x_i)$$

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$$\begin{aligned} E[X] =& 1P(X = 1) + 2P(X = 2) + 3P(X = 3) + 4P(X = 4) + 5P(X = 5) \\ =& 1\frac{2}{10} + 2\frac{4}{10} + 3\frac{1}{10} + 4\frac{1}{10} + 5\frac{2}{10} = \frac{27}{10} = 2.7 \end{aligned}$$

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- But also notice that

$$\begin{aligned} E[X] =& \frac{1}{10}\left(1 \cdot 2 + 2 \cdot 4 + 3 \cdot 1 + 4 \cdot 1 + 5 \cdot 2\right) \\ =& \frac{1}{10}\left(1 + 1 + 2 + 2 + 2 + 2 + 3 + 4 + 5 + 5\right) \end{aligned}$$

The Law of Large Numbers, again

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This is a generalization:

Theorem (The Law of Large Numbers)

Let X be a random variable whose value depends on a random experiment. Suppose the experiment is repeated n times and let $\bar{x}_n$ denote the arithmetic mean of the values of X in each trial. As n gets larger, the arithmetic mean $\bar{x}_n$ approaches the expected value $E[X]$ of that variable.

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